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Series expansions of the percolation probability on the directed triangular lattice

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Abstract. We have derived long-series expansions of the percolation probability for site, bond and site–bond percolation on the directed triangular lattice. For the bond problem we have extended the series from order 12 to 51 and for the site problem from order 12 to 35. For the site–bond problem, which has not been studied before, we have derived the series to order 32. Our estimates of the critical exponent β are in full agreement with results for similar problems on the square lattice, confirming expectations of universality. For the critical probability and exponent we find in the site case: $q_c = 0.404\,352\,8 \pm 0.000\,001\,0$ and $\beta = 0.276\,45 \pm 0.000\,10$; in the bond case: $q_c = 0.521\,98 \pm 0.000\,01$ and $\beta = 0.2769 \pm 0.0010$; and in the site–bond case: $q_c = 0.264\,173 \pm 0.000\,003$ and $\beta = 0.2766 \pm 0.0003$. In addition we have obtained accurate estimates for the critical amplitudes. In all cases we find that the leading correction to scaling term is analytic, i.e. the confluent exponent $\Delta = 1$.

1. Introduction

In an earlier paper (Jensen and Guttmann 1995) we reported on the derivation and analysis of long series for the percolation probability of site and bond percolation on the directed square and hexagonal lattices. In this paper we extend this work to site, bond and site–bond percolation on the directed triangular lattice. We refer to our earlier paper for a more general introduction to directed percolation and its role in the modelling of physical systems. In directed *site* percolation each site is either present (with probability p) or absent (with probability $q = 1 - p$) independent of all other sites on the lattice. Similarly for *bond* percolation each bond is absent or present independently of other bonds. Finally in *site–bond* percolation both sites and bonds may be absent or present with equal probability, but again with no dependency on any other sites or bonds. Two sites in the various models are connected if one can find a path, respecting the directions indicated in figure 1, through occupied sites, bonds or sites *and* bonds, respectively, from one to the other. When p is smaller than a critical value p_c all clusters of connected sites remain finite, while for $p \geqslant p_c$ there is an infinite cluster spanning the lattice in the preferred direction. The order parameter of the system is the percolation probability $P(p)$ that a given site belongs to the infinite cluster. This quantity is strictly zero when $p < p_c$ and changes continuously at p_c. For $p > p_c$ the behaviour of $P(p)$ in the vicinity of p_c may be described by a critical exponent β,

$$P(p) \propto (p - p_c)^{\beta} \qquad p \to p_c^+ . \tag{1}$$

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The bond problem was originally studied by Blease (1977) who calculated a series to 12th order. For the site problem De'Bell and Essam (1983) derived the series to 12th order. The site–bond problem has, at least to our knowledge, never been studied before. Our main motivation for doing so in this paper is to obtain further independent estimates of the critical exponent β. Using the finite-lattice method and the extrapolation technique of Baxter and Guttmann (1988) we have extended the series for the bond problem to order 51, for the site problem to order 35 and derived the series for the site–bond problem to order 32. The site and bond problems have also been studied by Essam *et al* (1986, 1988), who derived series expansions for moments of the pair connectedness.

2. The finite-lattice method

We wish to derive a series expansion for the percolation probability on the directed triangular lattice oriented as in figure 1. In this figure we have numbered the various levels or rows of the lattice according to which sites can be reached by a path of minimum length $N-1$ starting at the origin O. In other words all sites in the Nth row can be reached in $N-1$ steps but not in $N-2$ steps. Note that a path going through a given site can only reach the part of the lattice shown in figure 1 below the origin O. This suggests that one should look at the following finite-lattice approximation to $P(q)$, namely the probability $P_N(q)$ that the origin is connected to at least one site in the Nth row. Since we are in the high-density region we have chosen to use the expansion parameter q rather than p. $P_N(q)$ is a polynomial with integer coefficients and a maximal order determined by the total number of sites and/or bonds on the finite lattice.

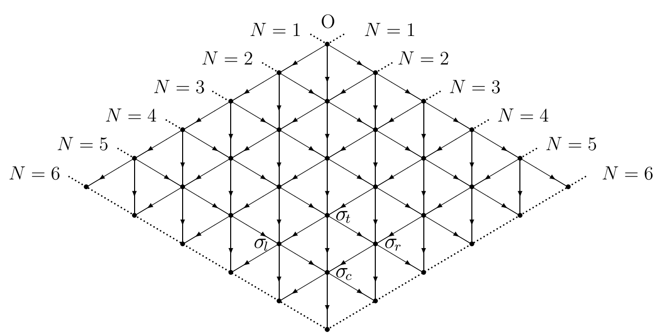

Figure 1. The directed triangular lattice with orientation given by the arrows. The rows are labelled according to the text.

By the method used for the square lattice problems (Bousquet-Mélou 1995) one can prove, *mutatis mutandis*, that the polynomials $P_N(q)$ converge to $P(q)$. Indeed we may consider $P(q) = \lim_{N\to\infty} P_N(q)$ to be a more precise definition of the percolation probability. More importantly, however, from a series expansion point of view, for the site and site–bond problems the first $N+1$ coefficients of the polynomials $P_N(q)$ are identical to those of $P(q)$. In the case of bond percolation the agreement extends through the first $2N+1$ coefficients.

2.1. Specification of the models

To calculate the finite-lattice percolation probability $P_N(q)$ we associate a state σ_j with each site, such that $\sigma_j = 1$ if site j is connected to the Nth row and $\sigma_j = -1$ otherwise. We shall often write $+/-$ for simplicity. Let l, c and r denote the sites connected to a site t from the row above, as in figure 1. We then define the weight function $W(\sigma_t|\sigma_l, \sigma_c, \sigma_r)$ as the probability that the top site t is in state σ_t, given that the lower sites l, c and r are in states σ_l, σ_c and σ_r, respectively. As for the square lattice (Bidaux and Forgacs 1984, Baxter and Guttmann 1988) we then have

$$P_N(q) = \sum_{\{\sigma\}} \prod_t W(\sigma_t|\sigma_l, \sigma_c, \sigma_r) \tag{2}$$

where the product is over all sites t of the lattice above the Nth row. The sum is over all values ± 1 of each σ_t, other than the topmost spin σ_1 which always takes the value $+1$. The spins in the Nth row are fixed at $+1$, and $P_N(q)$ is calculated as the sum over all possible configurations of the probability of each individual configuration.

The weight functions W are calculated as follows. Obviously, $W(-|\sigma_l, \sigma_c, \sigma_r) = 1 - W(+|\sigma_l, \sigma_c, \sigma_r)$. The remaining weights are easily calculated by considering the possible arrangements of states and sites and/or bonds. $W(+|-,-,-) = 0$ because the top site is connected to the Nth row if and only if at least one of its neighbours below is connected to the Nth row. All the remaining weights for the *site* problem equal $1-q$ because the top site has to be occupied in order to be connected to the Nth row. Let us next look at the remaining *bond* weights. $W^B(+|+,+,+) = 1 - q^3$ because the only bond configuration *not* allowed is all three bonds absent, which has probability q^3. $W^B(+|+,+,-) = W^B(+|+,-,+) = W^B(+|-,+,+) = 1 - q^2$ because the bond to the $-$ state can be either present or absent (probability 1) while among the remaining bonds only the configuration with both bonds absent (probability q^2) is forbidden. Finally, $W^B(+|+,-,-) = W^B(+|-,+,-) = W^B(+|-,-,+) = 1 - q$ because the bond to the + state has to be present, which happens with probability $p = 1 - q$, while the other bonds can be either present or absent. For the *site–bond* problem we find that $W^{SB}(+|\sigma_l, \sigma_c, \sigma_r) = (1-q)W^B(+|\sigma_l, \sigma_c, \sigma_r)$ because if the top state is $+1$ the top site has to be present.

2.2. Series-expansion algorithm

Computer algorithms for the calculation of $P_N(q)$ are readily found. These are basically implementations of the transfer matrix technique. The general features of these algorithms were described in our earlier paper (Jensen and Guttmann 1995), to which we refer for further details. The sum over configurations is performed by moving a boundary line through the lattice. For each configuration along the boundary line one maintains a (truncated) polynomial which equals the sum of the product of weights over all possible states on the side of the boundary already traversed. The boundary is moved through the lattice one site at a time. The calculation of $P_N(q)$ by this method is limited by memory, since one needs storage for 2^N boundary configurations. However, as was the case with the square lattice, this problem can be circumvented by introducing a cut into the lattice. For each fixed configuration of states on this cut one evaluates the lattice sum $P_N^C(q)$ and gets $P_N(q) = \sum_C P_N^C(q)$ as the sum over all configurations of the cut. By placing the cut appropriately, the growth in memory requirements can be reduced to $2^{N/2}$.

In figure 2 we show the triangular lattice with a cut marked by full circles. In the algorithm the cut is used as a pivot line by the boundary line which traverse the lattice. We

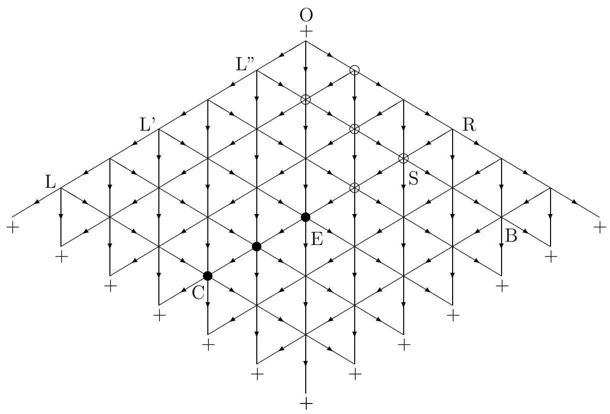

Figure 2. The directed triangular lattice with orientation given by the arrows. The sites with fixed states along the pivot line are marked by full circles. The open circles mark one particular position of the boundary line during the traversing of the lattice.

start by building up the first row at the base CL of the lattice. We then build up the part of the lattice above the cut from row CL to row EL′. Next the boundary line expands along the line-piece ES until it reach the position ESL″ and the last site (at L″) is flipped to the other site of the top-most triangle (after this the boundary line is in the position marked by the open circles). Then we work our way down the right-hand side of the lattice past R to position ESB. Finally the boundary line is moved down along the line-piece SEC after which the whole lattice has been build up. This process is then repeated for each configuration of the cut. Since the calculations for different cut-configurations are independent of each other this algorithm is perfectly suited to take full advantage of massively parallel computers.

Using this algorithm we calculated $P_N(q)$ for $N \leqslant 23$ for the bond and site–bond problems. The integer coefficients of $P_N(q)$ become very large so the calculation was performed using modular arithmetic (see, for example, Knuth 1969). Each run with $N = 23$, using a different moduli, took approximately 70 hours for the bond problem and 55 hours for the site–bond using 50 nodes on an Intel Paragon. For the site problem the weights only depend on whether or not there are any $+$'s among the neighbours of the top-most site. As was the case for the square site problem this may be used to sum over many configurations of the cut simultaneously (see Jensen and Guttmann 1995 for further details). This allowed us to calculate $P_N(q)$ for $N \leqslant 25$. Each run for $N = 25$ took about 85 hours using 50 nodes.

3. Extrapolation of the series

As mentioned, the coefficients of the polynomials $P_N(q) = \sum_{m\geqslant 0} a_{N,m} q^m$ will generally agree with those of the series for $P(q) = \sum_{m\geqslant 0} a_m q^m$ up to some order, $\tilde{N}$, determined by N, but depending on the specific problem. In the case of directed bond percolation on the square lattice Baxter and Guttmann (1988) found that the series for $P(q)$ could be extended significantly by determining correction terms to $P_N(q)$. Let us look at

$$P_N - P_{N+1} = q^{\tilde{N}} \sum_{r\geqslant 0} q^r d_{N,r} \qquad (3)$$

then we call $d_{N,r} = a_{N,\tilde{N}+r} - a_{N+1,\tilde{N}+r}$ the rth correction term. If formulae can be found for $d_{N,r}$ for all $r \leqslant K$ then, using the series coefficients of $P_N(q)$, one can extend the series for $P(q)$ to order $\tilde{N} + K$ since

$$a_{\tilde{N}+r} = a_{N,\tilde{N}+r} - \sum_{m=1}^{r} d_{N+r-m,m} \tag{4}$$

for all $r \leqslant K$. That this method can be very efficient was demonstrated by Baxter and Guttmann, who identified the first twelve correction terms for the square bond problem, and used $P_{29}(q)$ to extend the series for $P(q)$ to order 41. To really appreciate this advance one should bear in mind that the time it takes to calculate $P_N(q)$ grows exponentially with N, so a direct calculation correct to the same order would have taken years rather than days. In the following we will give details of the correction terms for the various directed percolation problems on the triangular lattice.

3.1. The site problem

For the site problem the coefficients of $P_N(q)$ agree with those of $P(q)$ to order N. In this case the first correction term is very simple as $d_{N,0} = 2$ for $N \geqslant 2$, i.e. the first correction term is simply a constant. For the second correction term $d_{N,1}$ we find the following sequence:

0, 0, 3, 18, 32, 50, 72, 98, ...

It is thus immediately clear that

$$d_{N,1} = 2N^2 \qquad \text{for} \quad N \geqslant 3\,. \tag{5}$$

Note that for convenience we assume that the sequence starts from $N = 0$. And indeed we find that for $N \geqslant r + 1$, $d_{N,r}$ can be expressed as a polynomial in N of order $2r$. We have been able to calculate these polynomials for the the first 10 correction terms. It turns out that it is useful to pull out a factor $1/(r!(r+1)!)$ and express the correction terms as

$$d_{N,r} = \frac{1}{r!(r+1)!} \sum_{k=0}^{2r} c_r^k N^k\,. \tag{6}$$

This ensures that the coefficients c_r^k in the extrapolation formulae are integers. We have listed these coefficients in table 1.

Obviously since these formulae are correct for $N \geqslant r + 1$ and we have calculated $P_N(q)$ for $N \leqslant 25$ we did not have enough terms in the correction sequences to calculate all the coefficients in these polynomials for the largest values of r. However, from the table of coefficients, it is immediately clear that $c_r^{2r} = 2^{r+1}$. And, in general, we found that $c_r^{2r-m}/2^{r+1}$ is a polynomial in r of order $2m$

$$c_r^{2r-m} = \frac{2^{r+1}}{(-4)^m m!} \sum_{k=0}^{2m} b_m^k r^k \tag{7}$$

where the prefactor has been chosen so as to make the leading coefficients particularly simple. In table 2 we have listed the coefficients b_m^k for the first six polynomials.

This time we note that $b_m^{2m} = 3^m$. And indeed as before we find that $b_m^{2m-j}/3^m$ is a polynomial in m of order $2j$. In particular, we have

$$b_m^{2m-1} = 3^m m(17/27 + 10/27m)$$

Table 1. The coefficients c_r^k in the extrapolation formulae (6) for the site problem.

	c_r^k								
k/r	1	2	3	4	5	6	7	8	9
0	0	24	0	5760	−345 600	−65 318 400	−15 850 598 400	−2 984 789 606 400	−539 895 767 040 000
1	0	−24	−48	−6720	662 400	86 728 320	15 417 077 760	3 039 204 188 160	681 914 690 150 400
2	4	4	160	−2256	−299 136	−54 616 320	−10 042 993 152	−2 801 552 624 640	−758 646 639 912 960
3		−12	−456	−5592	−155 040	29 156 640	6 930 400 512	1 683 396 497 664	492 391 103 938 560
4		8	112	6968	262 400	3 721 088	−1 895 857 152	−641 242 189 440	−236 796 916 234 752
5			−72	−4680	−211 440	−13 781 520	−275 292 864	183 056 948 928	80 349 078 951 936
6			16	1016	117 072	9 766 720	775 939 360	32 888 441 824	−13 942 053 553 664
7				−288	−35 760	−3 900 960	−484 442 784	−52 810 790 592	−3 002 221 192 320
8				32	6 000	1 183 584	180 360 704	27 746 932 192	4 062 978 111 936
9					−960	−222 000	−46 002 432	−9 468 263 616	−1 860 005 271 168
10					64	28 000	8 946 336	2 268 003 232	567 526 218 432
11						−2 880	−1 175 328	−405 615 168	−128 527 251 840
12						128	112 448	55 739 936	21 947 992 384
13							−8 064	−5 494 272	−2 918 143 872
14							256	406 784	301 743 168
15								−21 504	−23 270 400
16								512	1 362 432
17									−55 296
18									1 024

Table 2. The coefficients b_m^k in the extrapolation formulae (7) for the site problem.

	b_m^k					
k/m	1	2	3	4	5	6
1	3	$-3\frac{1}{3}$	192	$4\,662\frac{2}{5}$	−76 800	$2\,752\,914\frac{2}{7}$
2	3	19	−126	$-20\,702\frac{2}{3}$	−969 328	−61 888 160
3		$24\frac{2}{3}$	−411	7 092	1 554 956	$131\,279\,844\frac{4}{9}$
4		9	459	$21\,958\frac{1}{3}$	196 840	−55 417 284
5			141	$-17\,022\frac{2}{5}$	−1 359 655	$-81\,930\,639\frac{1}{3}$
6			27	$4\,615\frac{1}{3}$	860 155	105 874 935
7				684	−236 446	$-52\,835\,386\frac{20}{21}$
8				81	33 050	14 159 255
9					3 015	$-2\,180\,338\frac{4}{9}$
10					243	196 605
11						12 474
12						729

and

$$b_m^{2m-2} = 3^m m(1015/486 - 5137/1458m + 332/243m^2 + 50/729m^3) .$$

So when calculating the extrapolation formulae (6) we first used the sequences for the correction terms to predict as many polynomials as possible. When we ran out of terms we then predicted as many of the leading coefficients from (7) as possible. This, in turn, allowed us to find more extrapolation formulae, which we could use (together with the formulae for b_m^{2m-j}) to find more of the formulae for c_k^{2r-m}. And so on until the process stopped with the 10 extrapolation formulae we listed above.

Table 3. The coefficients a_n in the series expansion of the percolation probability $P(q)$ for directed site percolation on the triangular lattice.

n	a_n	n	a_n
0	1	18	−111 307
1	0	19	−255 236
2	0	20	−590 543
3	−1	21	−1 362 919
4	−2	22	−3 182 137
5	−5	23	−7 362 611
6	−10	24	−17 377 129
7	−20	25	−40 125 851
8	−41	26	−96 106 251
9	−86	27	−219 681 825
10	−182	28	−539 266 908
11	−393	29	−1 200 140 540
12	−853	30	−3 087 966 932
13	−1 887	31	−6 454 135 923
14	−4 208	32	−18 281 313 306
15	−9 445	33	−33 072 764 132
16	−21 350	34	−114 854 030 873
17	−48 612	35	−145 978 838 818

Using the ten extrapolation formulae and $P_{25}(q)$ we extended the series for $P(q)$ through order 35. The resulting series is listed in table 3.

3.2. *The site–bond problem*

For the site–bond problem the coefficients of $P_N(q)$ agree with those of $P(q)$ to order N. In this case the correction terms are very similar to those of the site problem. In particular we find that $d_{N,0} = 12$ and in general $d_{N,r}$ is a polynomial in N of order $2r$,

$$d_{N,r} = \frac{2^r}{r!(r+1)!} \sum_{k=0}^{2r} c_r^k N^k . \tag{8}$$

We have identified the first nine correction terms for the site–bond problem and have listed the coefficients c_r^k in the extrapolation formulae in table 4.

From this table it is immediately clear that the coefficient of the leading order $c_r^{2r} = 3 \times 4^r$. As in the site case we find that $c_r^{2r-m}/4^{r+1}$ is a polynomial in r of order $2m$.

$$c_r^{2r-m} = \frac{4^{r+1}}{(-4)^m m!} \sum_{k=0}^{2m} b_m^k r^k \tag{9}$$

where the prefactor has been chosen so as to make the leading coefficients particularly simple. In table 5 we have listed the coefficients b_m^k for the first six polynomials.

In this case $b_m^{2m} = 3^{m+1}$ and $b_m^{2m-1} = 3^{m+1} m(10/27 - 16/27m)$, which, using the same procedure as before allowed us to find the first 9 extrapolation formulae. From $P_{23}(q)$ we were thus able to extend the series for $P(q)$ through orde 32. The resulting series is listed in table 6.

Table 4. The coefficients c_r^k in the extrapolation formulae (8) for the site–bond problem.

	c_r^k							
k/r	1	2	3	4	5	6	7	8
0	−22	372	−6 948	228 960	−15 136 200	1 002 796 200	−148 319 942 400	16 196 987 318 400
1	−28	−88	−3 570	26 052	532 350	202 151 160	54 036 574 200	7 153 213 667 040
2	48	66	12 222	−66 190	16 300 863	−1 072 631 628	61 870 142 088	−28 771 509 693 672
3		−512	−6 804	−464 344	−9 400 240	−1 026 322 032	27 946 386 678	5 012 953 659 000
4		192	7 512	428 618	21 649 545	1 760 115 147	84 256 658 654	6 746 690 054 058
5			−4 800	−249 952	−23 384 790	−1 734 224 880	−194 249 017 018	−15 249 026 722 216
6			768	128 960	12 678 024	1 443 885 081	172 767 873 502	22 487 197 814 172
7				−34 816	−5 084 160	−762 064 416	−111 221 029 556	−18 388 293 899 920
8				3 072	1 447 680	274 270 176	53 077 387 932	10 265 902 430 946
9					−220 160	−72 890 880	−18 083 074 464	−4 339 851 543 328
10					12 288	13 020 672	4 539 617 152	1 389 887 209 152
11						−1 277 952	−833 487 872	−335 678 443 520
12						49 152	101 771 264	61 228 145 664
13							−6 995 968	−8 139 063 296
14							196 608	721 256 448
15								−36 700 160
16								786 432

Table 5. The coefficients b_m^k in the extrapolation formulae (9) for the site–bond problem.

	b_m^k				
k/m	1	2	3	4	5
1	−2	$-30\frac{1}{2}$	−177	$-3\,187\frac{4}{5}$	−179 760
2	9	$3\frac{1}{2}$	$198\frac{1}{2}$	$-3\,178\frac{1}{2}$	−101 540
3		−44	−252	3 962	$563\,989\frac{2}{3}$
4		27	$491\frac{1}{2}$	$8\,568\frac{1}{2}$	$-153\,182\frac{1}{2}$
5			−342	$-11\,196\frac{1}{5}$	$-381\,038\frac{1}{3}$
6			81	6 733	$401\,698\frac{1}{2}$
7				−1 944	$-199\,151\frac{1}{3}$
8				243	57 705
9					−9 450
10					729

3.3. The bond problem

For the bond problem the coefficients of $P_N(q)$ agree with those of $P(q)$ to order $2N$. In this case the first correction term is more complicated. For the first correction term $d_{N,0}$ we find the following sequence:

$$1, 3, 9, 27, 83, 263, 857, \ldots$$

which we have identified as

$$d_{N,0} = 2C_N - 1 \tag{10}$$

where $C_N = (2N)!/((N+1)!N!)$ are the Catalan numbers, which also occur in the correction terms for the square bond problem. In general we find that for $r \leqslant 4$ the correction terms

Table 6. The coefficients a_n in the series expansion of the percolation probability $P(q)$ for directed site–bond percolation on the triangular lattice.

n	a_n	n	a_n
0	1	17	−86 564 874
1	0	18	−134 834 422
2	0	19	−1 031 059 888
3	−8	20	−1 842 094 489
4	−4	21	−12 140 138 712
5	−70	22	−27 303 542 028
6	−23	23	−133 912 895 295
7	−640	24	−447 687 526 744
8	−205	25	−1 274 069 580 864
9	−6 272	26	−7 565 668 332 198
10	−2 941	27	−10 362 711 920 204
11	64 028	28	−113 855 530 577 726
12	−47 391	29	−131 148 651 484 930
13	−678 361	30	−1 188 175 707 628 214
14	−714 246	31	−4 485 228 802 915 811
15	−7 495 405	32	1 963 925 987 626 925
16	−10 059 661		

Table 7. The coefficients a_r^k, b_r^k and c_r^k in the extrapolation formulae (11) for the bond problem.

	a_r^k				b_r^k				c_r^k			
k/r	1	2	3	4	1	2	3	4	1	2	3	4
0									−1	−8	0	−2304
1	6	0	52	−418	2	−12	90	−748	2	12	108	1152
2	−4	−18	−56	88		2	−14	102	−1	−18	−176	−1112
3		10	72	288			2	−16		8	234	2392
4			−28	−284				2		−2	−125	−3526
5				84							36	2344
6											−5	−820
7												160
8												−14

are given, for $N \geqslant r-2$, by the formulae

$$d_{N,r} = \sum_{k=1}^{r+1} a_r^k C_{N+k-1} + \sum_{k=1}^{r} b_r^k \binom{N}{k} c_N + \frac{1}{r!r!} \sum_{k=0}^{2r} c_r^k N^k . \tag{11}$$

We have listed the coefficients a_r^k, b_r^k and c_r^k of these extrapolation formulae in table 7. We note that as in the previous problems the leading coefficients are quite simple, $a_r^{r+1} = (-1)^r 2C_{r+1}$, $b_r^r = 2$, and $c_r^{2r} = -C_r$.

These five extrapolation formulae and $P_{23}(q)$ allowed us to extend the series for $P(q)$ through order 51. The resulting series is listed in table 8.

4. Analysis of the series

We expect that the series for the percolation probability behaves like

$$P(q) \sim A(1-q/q_c)^{\beta}[1 + a_{\Delta}(1-q/q_c)^{\Delta} + \cdots] \tag{12}$$

Table 8. The coefficients a_n in the series expansion of the percolation probability $P(q)$ for directed bond percolation on the triangular lattice.

n	a_n	n	a_n
0	1	26	1 587 391
1	0	27	−3 535 398
2	0	28	6 108 103
3	−1	29	−13 373 929
4	0	30	23 438 144
5	−3	31	−50 592 067
6	1	32	89 703 467
7	−9	33	−191 306 745
8	6	34	342 473 589
9	−29	35	−722 890 515
10	27	36	1 304 446 379
11	−99	37	−2 729 084 244
12	112	38	4 957 423 139
13	−351	39	−10 292 036 449
14	450	40	18 800 279 417
15	−1 275	41	−38 769 381 587
16	1 782	42	71 154 482 443
17	−4 704	43	−145 869 275 322
18	6 998	44	268 798 182 822
19	−17 531	45	−548 189 750 051
20	27 324	46	1 013 680 069 047
21	−65 758	47	−2 057 857 140 279
22	106 211	48	3 816 820 768 061
23	−247 669	49	−7 717 195 669 953
24	411 291	50	14 352 037 073 232
25	−935 107	51	−28 915 083 150 931

where A is the critical amplitude, Δ the leading confluent exponent and the ... represents higher order correction terms. In the following sections we present the results of our analysis of the series which include accurate estimates for the critical parameters q_c, β, A and Δ. For the most part the best results are obtained using Dlog Padé (or in some cases just ordinary Padé) approximants. A comprehensive review of these and other techniques for series analysis may be found in Guttmann (1989).

4.1. q_c and β

In table 9 we list various Dlog Padé approximants to the percolation probability series for directed site percolation on the triangular lattice. The defective approximants, those for which there is a spurious singularity on the positive real axis closer to the origin than the physical critical point, are marked with an asterisk. Most higher-order approximants yield estimates around the values $q_c = 0.404\,352\,8$ and $\beta = 0.276\,45$, with very little spread among the approximants. Opting for a conservative error estimate, it seems appropriate to estimate that the critical parameters lie in the ranges, $q_c = 0.404\,352\,8(10)$ and $\beta = 0.276\,45(10)$, where the figures in parenthesis indicate the estimated error on the last digits.

The results of the analysis of the series for the bond problem are listed in table 10. In this case the spread among the various approximants is quite substantial, there appears to be a marked downward drift in the estimates for both q_c and β, and the estimates do not

Table 9. Dlog Padé approximants to the percolation series for directed site percolation on the triangular lattice.

N	$[N-1, N]$		$[N, N]$		$[N+1, N]$	
	q_c	β	q_c	β	q_c	β
5	0.404 092 8	0.274 51	0.403 461 0	0.270 45	0.404 523 6	0.278 22
6	0.403 850 0	0.273 01	0.407 425 1	0.313 68	0.404 877 5	0.281 15
7	0.404 378 7	0.276 71	0.404 333 1	0.276 33	0.404 367 7	0.276 64
8	0.404 353 5	0.276 51	0.404 380 3	0.276 76	0.404 369 8	0.276 66
9	0.404 361 5	0.276 58	0.404 363 6	0.276 60	0.404 355 5	0.276 50
10	0.404 362 3	0.276 58	0.404 358 2	0.276 54	0.404 357 4	0.276 53
11	0.404 356 7	0.276 52	0.404 356 7	0.276 52	0.404 357 6*	0.276 53*
12	0.404 356 7*	0.276 52*	0.404 361 0*	0.276 56*	0.404 355 3	0.276 50
13	0.404 352 5	0.276 44	0.404 353 8	0.276 47	0.404 358 0*	0.276 53*
14	0.404 352 9	0.276 45	0.404 352 6	0.276 45	0.404 352 8	0.276 45
15	0.404 352 7	0.276 45	0.404 352 9	0.276 45	0.404 352 8	0.276 45
16	0.404 352 8	0.276 45	0.404 352 8	0.276 45	0.404 352 8	0.276 45
17	0.404 352 8	0.276 45				

Table 10. Dlog Padé approximants to the percolation series for directed bond percolation on the triangular lattice.

N	$[N-1, N]$		$[N, N]$		$[N+1, N]$	
	q_c	β	q_c	β	q_c	β
10	0.522 223 5*	0.280 59*	0.524 191 8*	0.258 76*	0.522 085 3	0.278 98
11	0.522 183 5	0.280 19	0.522 107 8	0.279 27	0.522 095 8	0.279 12
12	0.522 069 1	0.278 73	0.522 038 8	0.278 23	0.521 836 6	0.272 95
13	0.522 133 6*	0.279 48*	0.521 968 0	0.276 78	0.522 284 4*	0.280 38*
14	0.522 027 8	0.278 05	0.522 002 9	0.277 55	0.522 008 6	0.277 68
15	0.522 007 6	0.277 65	0.522 006 4	0.277 63	0.521 997 3	0.277 41
16	0.522 010 1*	0.277 70*	0.521 961 3	0.276 16	0.521 994 2	0.277 33
17	0.522 004 6	0.277 59	0.521 989 5	0.277 20	0.521 995 9*	0.277 38*
18	0.522 077 4*	0.277 68*	0.521 833 5	0.266 12	0.521 977 0	0.276 79
19	0.522 038 2*	0.278 01*	0.521 994 4	0.277 35	0.521 987 6	0.277 15
20	0.521 979 5	0.276 87	0.521 984 8	0.277 06	0.521 984 6	0.277 05
21	0.521 984 6	0.277 05	0.521 984 8	0.277 05	0.521 984 7	0.277 05
22	0.521 984 7	0.277 05	0.521 984 8*	0.277 05*	0.521 978 0	0.276 78
23	0.521 983 7	0.277 02	0.521 982 0	0.276 96	0.521 981 1	0.276 92
24	0.521 976 7	0.276 71	0.521 980 4	0.276 89	0.521 983 0*	0.276 99*
25	0.521 979 6	0.276 86	0.521 982 7*	0.276 98*		

settle down to definite values. It does, however, seem likely that the true critical parameters lie within the estimates: $q_c = 0.521\,98(1)$ and $\beta = 0.2769(10)$.

The analysis of the series for the site–bond problem yields the results in table 11. Again we see a downward drift in the estimates for both q_c and β though the estimates are somewhat more stable than in the previous case. We estimate that the true critical parameters lie within the ranges: $q_c = 0.264\,173(3)$ and $\beta = 0.2766(3)$

Table 11. Dlog Padé approximants to the percolation series for directed site–bond percolation on the triangular lattice.

N	$[N-1, N]$		$[N, N]$		$[N+1, N]$	
	q_c	β	q_c	β	q_c	β
5	0.263 955 2	0.274 56	0.263 977 5	0.274 75	0.264 506 6	0.280 77
6	0.264 784 6	0.285 59	0.264 075 3	0.275 56	0.264 162 2	0.276 47
7	0.264 169 5	0.276 56	0.264 149 4	0.276 32	0.264 156 0	0.276 40
8	0.264 157 6	0.276 42	0.264 247 6	0.278 35	0.264 166 7	0.276 54
9	0.264 167 9	0.276 55	0.264 173 9	0.276 65	0.264 174 7	0.276 66
10	0.264 174 7	0.276 66	0.264 173 4*	0.276 64*	0.264 175 7	0.276 68
11	0.264 175 8	0.276 68	0.264 175 3	0.276 67	0.264 175 4	0.276 67
12	0.264 175 4	0.276 67	0.264 175 3	0.276 67	0.264 175 5*	0.276 68*
13	0.264 175 5*	0.276 68*	0.264 175 4*	0.276 68*	0.264 175 5*	0.276 68*
14	0.264 175 5*	0.276 68*	0.264 175 0	0.276 67	0.264 171 6	0.276 54
15	0.264 172 4	0.276 58	0.264 173 5	0.276 63	0.264 172 6	0.276 59
16	0.264 172 9	0.276 60				

4.2. The critical amplitudes

We can estimate the critical amplitude A by evaluating Padé approximants to $G(q) = (q_c - q)P^{-1/\beta}$ at q_c, since it follows from the leading critical behaviour in (12) that $G(q_c) \sim A^{-1/\beta} q_c$. This procedure works well but requires knowledge of both q_c and β. As we have just shown, we know both q_c and β very accurately for the triangular site series. We estimated A using values of q_c between 0.404 352 4 and 0.404 353 4 and values of β ranging from 0.2764 to 0.2765. For each (q_c, β) pair we calculate A as the average over all $[N+K, N]$ Padé approximants with $K = 0, \pm 1$ and $2N + K \geqslant 25$. The spread among the approximants is minimal for $q_c = 0.404\,352\,7$, $\beta = 0.276\,45$ where $A = 1.581\,883(5)$. Allowing for values of q_c and β within the full range we get $A = 1.5819(4)$.

For the bond problem we used values of q_c from 0.521 96 to 0.521 21 and β from 0.2763 to 0.2773 averaging over Padé approximants with $2N + K \geqslant 40$. In this case the spread is minimal for $q_c = 0.521\,985$, $\beta = 0.2767$ where $A = 1.485\,84(2)$. Again allowing for a wider choice of critical parameters we estimate that $A = 1.486(6)$.

For the site–bond series we restricted q_c to lie between 0.264 170 and 0.264 176 and β between 0.2763 to 0.2768 using all approximants with $2N + K \geqslant 25$. The minimal spread occurs at $q_c = 0.264\,173$, $\beta = 0.2766$ where $A = 1.477\,393(4)$. A wider choice for q_c and β leads to the estimate $A = 1.477(1)$.

4.3. The confluent exponent

We studied the series using two different methods in order to estimate the value of the confluent exponent. In the first method, due to Baker and Hunter (1973), one transforms the function P,

$$P(q) = \sum_{i=1}^{n} A_i (1 - q/q_c)^{-\lambda_i} = \sum_{n=0}^{\infty} a_n q^n \qquad (13)$$

into an auxiliary function with simple poles at $1/\lambda_i$. We first make the change of variable $q = q_c(1-\mathrm{e}^{-\zeta})$ and find, after multiplying the coefficient of ζ^k by $k!$, the auxiliary function

$$\mathcal{F}(\zeta) = \sum_{i=1}^{N}\sum_{k=0}^{\infty} A_i(\lambda_i\zeta)^k = \sum_{i=1}^{N}\frac{A_i}{1-\lambda_i\zeta} \tag{14}$$

which has poles at $\zeta = 1/\lambda_i$ with residue $-A_i/\lambda_i$. The great advantage of this method is that one obtains simultaneous estimates for many critical parameters, namely, β (the dominant singularity), Δ (the sub-dominant singularity), and the critical amplitudes (the residues at the singularities), while there is only one parameter q_c in the transformation. Unfortunately this method does not appear to work well for this problem. For the site problem we find that the transformed series generally yields poor estimates for β and no estimates for the confluent exponent. For the bond and site–bond problem the situation is somewhat better. In table 12 we have listed estimates for the critical parameters obtained from various Padé approximants to the Baker-Hunter transformed series, using the values $q_c = 0.521\,98$ for the bond series and $q_c = 0.264\,173$ for the site–bond series.

Table 12. The critical exponent β, confluent exponent Δ and critical amplitudes A and a_Δ obtained from $[N, M]$ Padé approximants to the Baker-Hunter transformed series for the bond and site–bond problems.

N	M	β	A	Δ	$A \times a_\Delta$
Bond problem					
18	19	0.276 62	1.484 69	1.038 97	2.216 46
19	20	0.277 05	1.488 45	0.971 24	1.813 01
20	21	0.276 78	1.486 04	1.013 27	2.044 00
21	21	0.280 38	1.498 43	0.911 20	1.685 64
21	22	0.276 77	1.485 94	1.015 30	2.056 71
22	22	0.276 73	1.485 82	1.016 56	2.062 89
22	23	0.276 77	1.485 94	1.015 30	2.056 72
23	23	0.275 59	1.482 08	1.064 73	2.347 14
23	24	0.276 76	1.485 87	1.016 57	2.064 77
24	25	0.276 80	1.486 19	1.010 64	2.027 88
25	26	0.276 79	1.486 15	1.011 33	2.032 11
Site–bond problem					
11	12	0.277 88	1.487 49	0.898 58	1.621 93
12	13	0.276 51	1.476 68	1.010 68	2.168 27
13	13	0.273 42	1.469 40	1.113 95	3.151 55
13	14	0.276 51	1.476 66	1.010 91	2.169 97
14	15	0.276 61	1.477 45	0.999 50	2.089 54
15	15	0.278 28	1.481 82	0.960 56	1.910 13
15	16	0.276 59	1.477 28	1.001 94	2.106 41

It should be noted that, obviously, all approximants yield estimates for the critical parameters. However, we have discarded many approximants from the table because we believe the results to be spurious. For all the discarded approximants we found that the amplitude of the confluent term was of order zero and generally the estimate for β was very far from the expected value. Among the remaining approximants we clearly see that the favoured value of the confluent exponent is $\Delta = 1$. We also note that the amplitude estimates are in full agreement with those of the previous section.

In the second method, due to Adler *et al* (1981), one studies Dlog Padé approximants to the function $F(q)$, where

$$F(q) = \beta P(q) + (q_c - q)\mathrm{d}P(q)/\mathrm{d}q\,.$$

The logarithmic derivative of $F(q)$ has a pole at q_c with residue $\beta + \Delta$. We evaluate the Dlog Padé approximants for a range of values of q_c and β. In table 13 we have listed the estimates for Δ obtained by averaging over all $[N, N+K]$ approximants for a few values of β with q_c fixed at the central value of our estimate range. For the site and site–bond problem we used all approximants with $2N + K \geqslant 25$ and for the bond problem all approximants with $2N + K \geqslant 40$. This analysis clearly indicates that $\Delta \simeq 1$ and thus that there is no sign of any non-analytic corrections to scaling.

Table 13. Estimates for the confluent exponent Δ from the transformation due to Adler *et al* (1981) for various values of β at the critical point q_c.

Site problem		Site–bond problem		Bond problem	
β	Δ	β	Δ	β	Δ
0.276 40	0.985 87	0.276 30	0.970 76	0.276 60	1.034 71
0.276 41	0.990 03	0.276 35	0.982 20	0.276 65	1.030 79
0.276 42	0.993 78	0.276 40	0.991 36	0.276 70	1.025 37
0.276 43	0.996 83	0.276 45	0.997 96	0.276 75	1.018 46
0.276 44	0.998 90	0.276 50	1.001 76	0.276 80	1.010 13
0.276 45	0.999 79	0.276 55	1.002 62	0.276 85	1.000 42
0.276 46	0.999 42	0.276 60	1.000 47	0.276 90	0.989 41
0.276 47	0.997 82	0.276 65	0.995 33	0.276 95	0.977 16
0.276 48	0.995 14	0.276 70	0.987 32	0.277 00	0.963 77
0.276 49	0.991 64	0.276 75	0.976 63	0.277 05	0.949 34
0.276 50	0.987 55	0.276 80	0.963 52	0.277 10	0.933 94

5. Conclusion

In this paper we have presented extended series for the percolation probability for site, bond and site–bond percolation on the directed triangular lattice. The analysis of the series leads to improved estimates for the percolation threshold and the order parameter exponent β. In table 14 we summarize the critical parameter estimates for the percolation probability for the three problems on the triangular lattice studied here and the problems studied in our earlier paper. The estimates for $q_c = 1 - p_c$ for the triangular bond and site problems are in excellent agreement with those obtained by Essam *et al* (1986, 1988), $q_c = 0.404\,37(7)$ and $q_c = 0.521\,975(7)$, respectively. The estimates for β clearly show, as one would expect, that all the models studied in this and our earlier paper belong to the same universality class. The unbiased estimates for β, derived in the manner described in the previous section, for the triangular site and square bond cases are in excellent agreement and have small error bars (we emphasize once more that our error estimates are conservative). This leads us to believe that an improved estimate $\beta = 0.276\,44(3)$ is reasonable. We used this highly accurate estimate to obtain the *biased* estimates in table 14 as follows. First we formed the series for $P(q)^{-1/\beta}$ using $\beta = 0.276\,44$. This series has a simple pole at q_c which can be estimated from ordinary Padé approximants. By averaging over all $[N, N+K]$ approximants with $K = 0, \pm 1$ and $2N + K \geqslant N_{\mathrm{min}}$ we obtained the biased estimates for q_c the error bars are basically twice the spread among the approximants. We then used the biased estimate for

q_c (with β as before) to obtain the biased estimates for the amplitudes using the procedure described in the previous section. As previously noted (Jensen and Guttmann 1995), there is no simple rational fraction whose decimal expansion agrees with our estimate of β. Given that this model is not conformally invariant, and that the expectation of exponent rationality is a consequence of conformal invariance, it is perhaps naive to expect otherwise. It is nevertheless true that there is a widely held—if imprecisely expressed—view that two dimensional systems should have rational exponents. More precise numerical work such as the recent estimation of the longitudinal size exponent $\nu_{||}$ (Conway and Guttmann 1994) of directed animals and the present calculation, supports the conclusion that the critical exponents for these models should not be expected to be simple rational fractions. Finally note that none of the series show any evidence of non-analytic confluent correction terms. This provides a hint that the models might be exactly solvable.

Table 14. Estimates of critical parameters for the three problems on the triangular (T) lattice studied in this paper and for the site and bond problems on the directed square (S) and honeycomb (H) lattices. See the text for explanation of the biased estimates.

	Unbiased estimates			Biased estimates		
Problem	q_c	β	A	q_c	A	$N_{\min}$
T bond	0.521 98(1)	0.2769(10)	1.486(6)	0.521 971(5)	1.4841(2)	45
T site	0.404 352 8(10)	0.276 45(10)	1.5819(5)	0.404 352 3(3)	1.581 83(2)	30
T site–bond	0.264 173(3)	0.2766(3)	1.477(1)	0.264 170(4)	1.4765(3)	25
S bond	0.355 299 4(10)	0.276 43(10)	1.3292(5)	0.355 299 55(15)	1.329 20(1)	45
S site	0.294 515(5)	0.2763(3)	1.425(1)	0.294 518(3)	1.425 88(4)	30
H bond	0.177 143(2)	0.2763(2)	1.106(1)	0.177 144(2)	1.1064(3)	30
H site	0.160 067(5)	0.2763(4)	1.167(1)	0.160 069(2)	1.1680(3)	30

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Appendix A. The first extrapolation formulae

In this appendix we shall calculate the first correction term(s) $d_{N,r}$ for the various problems we have studied in this paper. In the following we rely heavily on the work of Bousquet-Mélou (1995) and we shall represent the directed percolation models in terms of directed animals. By a directed site (bond) animal A we simply understand any finite set of connected sites (bonds) starting at the origin O in figure 1. The *area* (or size) $|A|$ of an animal is the number of sites in the animal and the *perimeter* $p(A)$ is the number of unoccupied sites (bonds) with a nearest neighbour in A. The *height* h of an animal is the last row to which the animal extends, i.e. there is at least one occupied site in row h belonging to A but none in row $h+1$. The percolation probability, for the site and site–bond cases, is

$$P(q) = 1 - \sum_{A \in \mathcal{A}} q^{p(A)} (1-q)^{|A|-1} \tag{A1}$$

where $\mathcal{A}$ denotes the set of animals on the lattice. For bond percolation the power of $(1-q)$ in the above equation is $|A|$. The difference stems from the assumption that for site percolation the origin is occupied with probability 1. In analogy with the finite-lattice formulation we define subsets $\mathcal{A}_N$ of $\mathcal{A}$ as the set of animals of height less than N. It follows that

$$P_N(q) = 1 - \sum_{A\in\mathcal{A}_N} q^{p(A)}(1-q)^{|A|-1} \tag{A2}$$

and

$$P_N(q) - P_{N+1}(q) = 1 - \sum_{A\in\mathcal{A}_{N+1}\backslash\mathcal{A}_N} q^{p(A)}(1-q)^{|A|-1}\,. \tag{A3}$$

It should be noted that in the site and site–bond cases the polynomials $P_N(q)$ defined above are identical to the polynomials $P_{N+1}(q)$ from section 2. From (A3) it is immediately clear that P_N and P_{N+1} agree up to an order $\tilde{N}$ determined by the animals in $\mathcal{A}_{N+1}\backslash\mathcal{A}_N$ with the smallest perimeter. In our cases $\tilde{N}$ is simply proportional to N and the polynomials $P_N(q)$ therefore have a formal limit $P_\infty(q)$ which we identify as the percolation probability $P(q)$. By expanding (A3) one gets a very useful expression for the correction terms

$$P_N(q) - P_{N+1}(q) = q^{\tilde{N}}\sum_{r\geqslant 0} q^r d_{N,r} = q^{\tilde{N}}\sum_{r\geqslant 0} q^r \sum_{k=0}^{r}\sum_{\mathcal{A}\in\mathcal{A}_{N,k}} (-1)^{r-k}\binom{|A|-1}{r-k} \tag{A4}$$

where $\mathcal{A}_{N,k} = \{A \in \mathcal{A}_{N+1}\backslash\mathcal{A}_N,\ \ p(A) = \tilde{N} + k\}$.

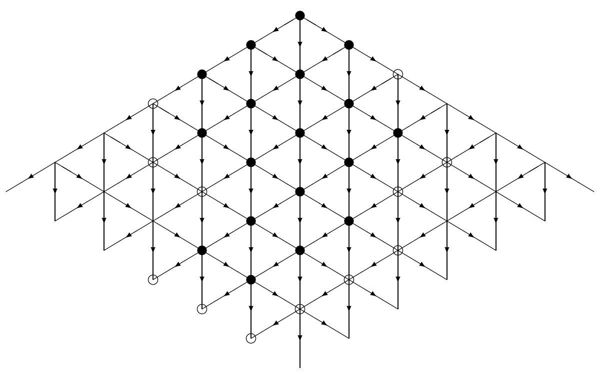

Figure A1. A compact directed site animal (filled circles) on the triangular lattice with perimeter sites marked by open circles.

The site case

An animal is *compact* if the occupied sites in any given row are consecutive, i.e. there are no holes in the animal (see figure A1). Obviously, removing interior sites from a compact animal can never reduce the perimeter. Therefore, the animals in $\mathcal{A}_{N+1}\backslash\mathcal{A}_N$ with minimal perimeter are compact. The minimal perimeter of a compact animal of height N is $N+2$. This is proved by induction on N. It is obviously true for $N = 1$ and one can easily see that by adding sites in row $N+1$ to a compact animal of height N at least one more perimeter-site is added. We also note that there are at least two animals of height N with perimeter $N+2$, namely a string of sites (one per row) running down either the left or right hand side of the lattice. This shows that $\tilde{N} = N+2$. It is also clear that these two animals must be the ones that give rise to the first correction term $d_{N,0} = 2$. What remains is to prove that there can be no more animals in $\mathcal{A}_{N+1}\backslash\mathcal{A}_N$ with perimeter $N+2$. In order to do this we need a unique way of characterizing the perimeter of compact animals of height N. Introduce lines R_k (L_k) parallel to the right-hand (left-hand) edge starting from row k. Since the animal is compact all sites in A intersecting R_k and L_k are consecutive. The number of perimeter sites on the left-hand side of the animal is $w_l = \max\{k, L_k \cap A \neq \emptyset\}$ because the last occupied site in line L_k has an unoccupied neighbour on L_k. Similar arguments apply for the number of perimeter sites w_r on the right-hand side of A. Finally, we note that the only perimeter site not accounted for is the one lying vertically below the last site in L_N and/or R_N. So the perimeter is $p(A) = w_l + w_r + 1$. Furthermore, if $A \in \mathcal{A}_{N+1} \backslash \mathcal{A}_N$ then either w_l or w_r (possibly both) has to equal N. The animals with minimal perimeter $N+2$ are those with $w_l = 1$ or $w_r = 1$, obviously there can be only two such animals, which completes our proof that $d_{N,0} = 2$.

From equation (A4) we get the second correction term

$$d_{N,1} = |\mathcal{A}_{N,1}| - \sum_{A \in \mathcal{A}_{N,0}} (|A| - 1) \tag{A5}$$

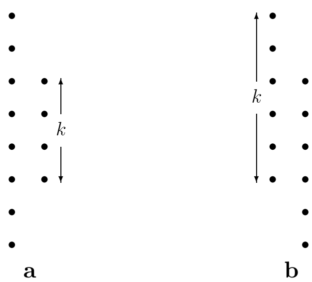

Figure A2. The two types of compact directed site animals with $w_l = 2$ which contribute to the second correction term.

where $|\mathcal{A}_{N,1}|$ is the number of animals of height N with a perimeter of length $N+3$. From the characterization of compact animals derived above it follows that the animals in $\mathcal{A}_{N,1}$ are those with $w_l = 2$ or $w_r = 2$. Obviously there is the same number of animals in each case so we can restrict ourselves to the case $w_l = 2$, $w_r = N$. We are thus looking at animals restricted to the left-most two lines L_1 and L_2 of the lattice and either $L_1 \cap R_N$ or $L_2 \cap R_N$ has to be non-empty. The two types of animals are illustrated in figure A2. If $L_1 \cap R_N \neq \emptyset$ (figure A2(a)) then the first N sites of L_1 are occupied and $1 \leqslant k \leqslant N$ consecutive sites along L_2 are occupied; these k sites can be placed in $N-k+1$ positions. If $L_1 \cap R_N = \emptyset$ (figure A2(b)) and the first k sites ($1 \leqslant k \leqslant N-1$) of L_1 are occupied then the first j consecutive sites $0 \leqslant j \leqslant k$ of L_2 may be empty. Combining these two contributions with those from $w_r = 2$ we find

$$|\mathcal{A}_{N,1}| = 2\left(\sum_{k=1}^{N}(N-k+1) + \sum_{k=1}^{N-1}(k+1)\right) = 2N^2 + 2N - 2\,.$$

Since the number of sites in each of the two animals in $\mathcal{A}_{N,0}$ is N, equation (A5) yields

$$d_{N,1} = 2N^2$$

thus proving the empirical formula derived previously.

Next we prove the formula for $d_{N,2}$. From equation (A4) we see that the third correction term is given by

$$d_{N,2} = |\mathcal{A}_{N,2}| - \sum_{A \in \mathcal{A}_{N,1}} (|A| - 1) + \sum_{A \in \mathcal{A}_{N,0}} \binom{|A| - 1}{2}\,. \tag{A6}$$

In this case there are two distinctly different sets of animals in $\mathcal{A}_{N,2}$, namely, compact animals with $w_l = 3$ as pictured in figure A3, and animals formed from the compact animals of figure A2 by removing consecutive sites from the second line of occupied sites leaving at least the first and last sites untouched. One easily sees that cutting such a 'hole' in these animals is the only way of increasing their perimeter by one site. From the animals in figure A3 we get the following contributions:

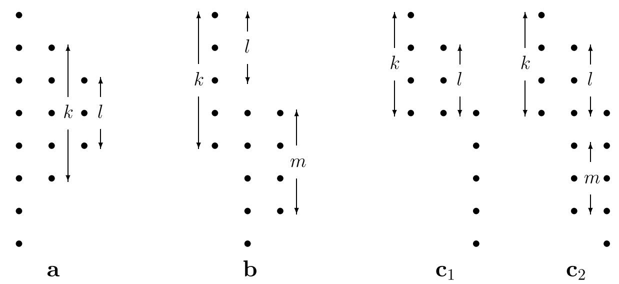

Figure A3. The types of compact directed site animals with $w_l = 3$ which contributes to the third correction term.

$$\begin{aligned}
a: &\quad 2\sum_{k=1}^{N}\sum_{l=1}^{k}(N-k+1)(k-l+1) = \tfrac{1}{12}N^4 + \tfrac{1}{2}N^3 + \tfrac{11}{12}N^2 + \tfrac{1}{2}N \\
b: &\quad 2\sum_{k=1}^{N-1}\sum_{l=1}^{k}\sum_{m=1}^{N-l}(N-l-m+1) = \tfrac{1}{4}N^4 + \tfrac{5}{6}N^3 - \tfrac{1}{4}N^2 - \tfrac{5}{6}N \\
c_1: &\quad 2\sum_{k=1}^{N-1}\sum_{l=1}^{k}(l+1) = \tfrac{1}{3}N^3 + N^2 - \tfrac{4}{3}N \\
c_2: &\quad 2\sum_{k=1}^{N-2}\sum_{l=0}^{k}\sum_{m=1}^{N-k-1}(m+l+1) = \tfrac{1}{6}N^4 + \tfrac{1}{3}N^3 - \tfrac{7}{6}N^2 - \tfrac{4}{3}N + 2\,.
\end{aligned} \tag{A7}$$

The animals in figure A3(a) account for animals with $L_1 \cap R_N \neq \emptyset$, those of figure A3($b$) for animals with $L_1 \cap R_N = \emptyset$ and $L_2 \cap R_N \neq \emptyset$, and lastly those of figure A3(c) for animals where $L_1 \cap R_N = \emptyset$ and $L_2 \cap R_N = \emptyset$. The contribution in each case is simply all the possible configurations which leads to an animal of the specified kind. The sums in (A7) should be self-evident.

The animals in figure A4 with a cut as described above yield the contributions

$$\begin{aligned}
a: &\quad 2\sum_{k=3}^{N}\sum_{l=1}^{k-2}(N-k+1)(k-l-1) = \tfrac{1}{12}N^4 - \tfrac{1}{6}N^3 - \tfrac{1}{12}N^2 + \tfrac{1}{6}N \\
b: &\quad 2\sum_{k=2}^{N-1}\sum_{l=0}^{k-2}\sum_{m=1}^{k-l-1}(k-l-m) = \tfrac{1}{12}N^4 - \tfrac{1}{6}N^3 - \tfrac{1}{12}N^2 + \tfrac{1}{6}N\,.
\end{aligned} \tag{A8}$$

In case (a) the piece in the second line has to have at least three sites ($k \geqslant 3$) otherwise one could not cut out a hole of size $l \leqslant k-2$. The k sites can be placed in $(N-k+1)$ positions and the hole can be cut in $k-2-l+1 = k-l-1$ places, which leads to the first sum. In case (b) there can be from 2 to $N-1$ sites in the first line (the sum over k) with an overlap of $0 \leqslant m \leqslant k-2$ sites between the first line and the consecutive sites in the second line extending to the Nth row. Among the remaining $k-m$ sites in the second line $1 \leqslant l \leqslant k - m - 1$ are occupied and they can be placed in $k - m - l$ positions, thus giving us the second sum.

The second term in (A6) is the sum over $|A| - 1$ of the compact animals in figure A2 and we find the two contributions:

$$a: \quad 2\sum_{k=1}^{N}(N-k+1)(N+k-1) = \tfrac{4}{3}N^3 + N^2 - \tfrac{1}{3}N$$
$$b: \quad 2\sum_{k=1}^{N-1}\sum_{l=0}^{k}(N+k-l-1) = \tfrac{4}{3}N^3 - \tfrac{10}{3}N + 2\,. \qquad \text{(A9)}$$

Finally the last term in (A6) simply stems from the two animals in $\mathcal{A}_{N,0}$ and their contribution is

$$2\binom{N-1}{2} = N^2 - 3N + 2\,. \qquad \text{(A10)}$$

By adding the contributions of (A7), (A8) and (A10) while subtracting those of (A9) we get

$$d_{N,2} = \tfrac{2}{3}N^4 - N^3 + \tfrac{1}{3}N^2 - 2N + 2 = \tfrac{1}{12}(8N^4 - 12N^3 + 4N^2 - 24N + 24) \qquad \text{(A11)}$$

in full agreement with the extrapolation formula listed in table 1, thus concluding the proof for $d_{N,2}$.

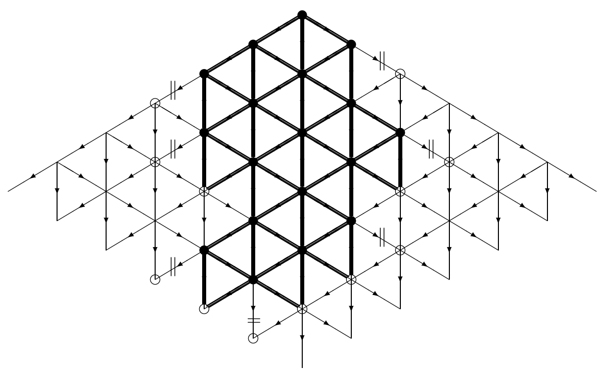

Figure A4. A site-compact directed site–bond animal (full circles and thick bonds) on the triangular lattice with possible perimeter sites marked by open circles. Some of the perimeter sites have only one possible incident bond (marked by double lines) and in those cases the bond can be present (the site is part of the perimeter) or absent (the edge is part of the perimeter).

The site–bond case

From the empirical extrapolation formulae it it clear that the site–bond case is very similar to the site case and only a few generalizations are necessary. Again we look at compact animals and the ones we shall call *site-compact* have the minimal perimeter. A site-compact animal is one in which, as before, all occupied sites and bonds in a row are consecutive and in addition *all possible bonds to sites with more than one incident edge are present*. Figure A4 shows such an animal. Clearly the perimeter of such an animal is equal to the perimeter of the identical *site* animal. Thus the animals with minimal perimeter have $w_l = 1$ (or $w_r = 1$). Such animals consist of consecutive occupied sites down the left-hand side with most of the bonds emanating from these sites present. A few of the bonds can be either present or absent, namely, the bond from the top site pointing South-East and the bonds from the last site pointing South-West or South, though in this latter case at least one of the bonds has to be present. So all in all there are three possible bond configurations from the last site and two from the top site for a total of six possibilities. Taking into account the animals with $w_r = 1$ we have proved

$$d_{N,0} = 12 .$$

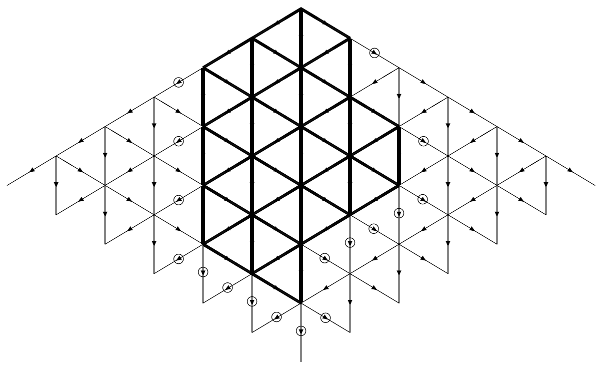

Figure A5. A compact directed bond animal (thick bonds) on the triangular lattice with perimeter bonds marked by open circles.

The bond case

The first correction term for the bond case, $d_{N,0} = 2C_N - 1$, involve the Catalan numbers C_N which equal the first correction term for the square bond problem (Baxter and Guttmann 1988). Bousquet-Mélou (1995) proved this result by noting that the square bond correction term arise from compact bond animals of directed height N. The first correction term for the triangular bond problem can be found by generalizing the arguments from the square bond case. The first correction arise from compact animals constructed as follows. Choose two paths ω_1 and ω_2 consisting of bonds pointing only South and South-West starting from the origin and terminating at the same point on level N. The animal obtained by filling in all bonds between ω_1 and ω_2 has height N and perimeter $2N + 1$. These animals are just the *staircase animals* which are enumerated by the Catalan numbers and give rise to the first square bond correction term. Obviously the set of animals bounded by paths consisting of South and South-East bonds also contribute to the first correction term. The animal consisting entirely of south bonds (a line of bonds down the centre of the lattice) is the only animal included in both sets. The first correction term is exactly due to these $2C_N - 1$ 'staircase animals' on the triangular lattice.

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